Approximate Matrix Rank

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- Represents the number of import components in A
- If the rank r is small, can greatly compress data:

$$\underbrace{\begin{bmatrix} \quad & & \\ & A & \\ & & \quad \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} \\ B \\ \\ \end{bmatrix}}_{m \times r} \underbrace{\begin{bmatrix} \quad C \quad \end{bmatrix}}_{r \times n}$$

What if there is noise?

$$\underset{m\times n}{\left[\quad \hat{A} \quad\right]} = \underset{m\times r}{\left[\, B \,\right]} \left[\; \underset{r\times n}{C} \;\right] + \underset{m\times n}{\left[\quad N \quad\right]}$$

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- Gaussian noise matrix N is full rank

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- The noisy data $\hat{A} = A + N$ is also full rank
- Don’t know the number of features in your data 🙁

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$$\mathrm{rank}_\varepsilon(A) = \min_{\|E\| \le \varepsilon} \mathrm{rank}(A + E)$$

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- A.K.A. numerical or approximate rank

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- Estimate standard Gaussian[1] $\|N\| \approx \sqrt{m} + \sqrt{n}$

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Illustration of ε-rank

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- Generate random 9×9 rank-4 matrix A

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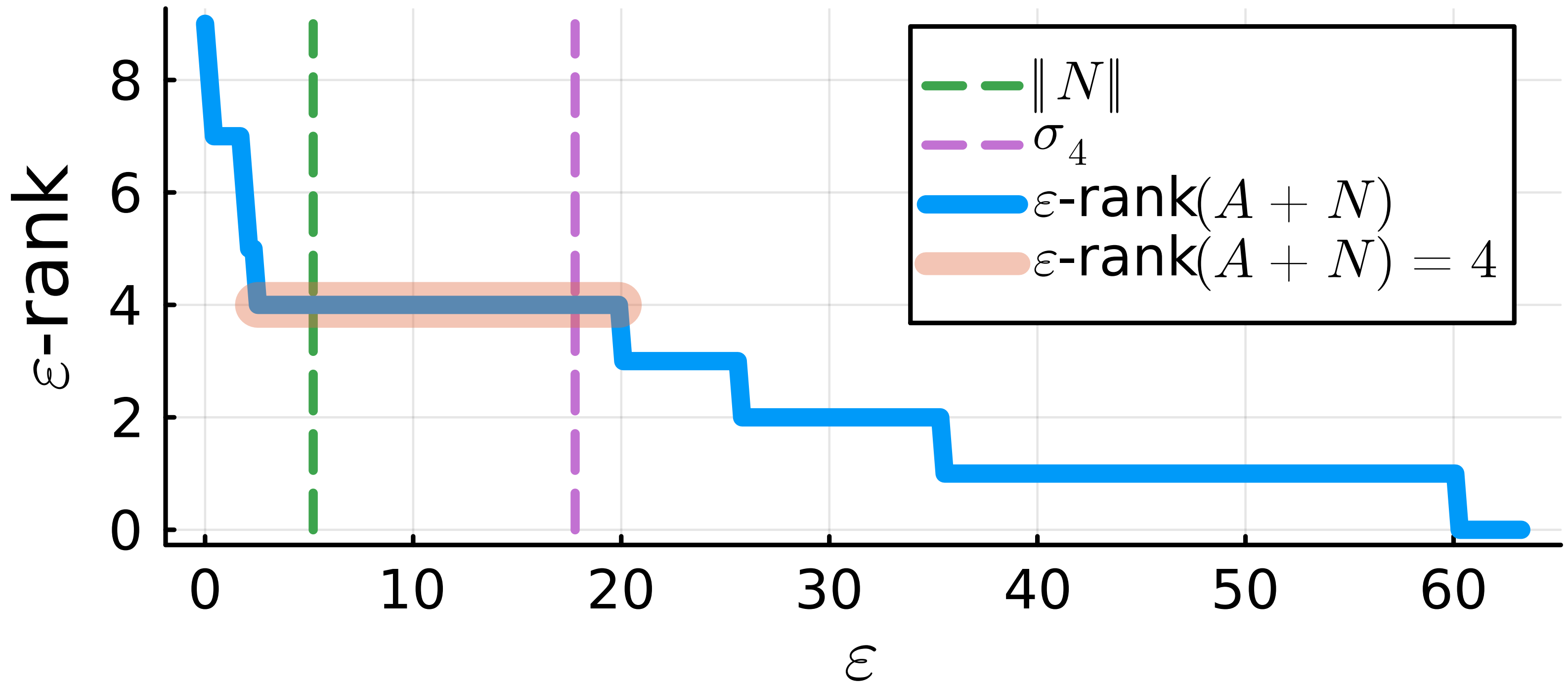

Future Work & References

Golub, Gene H., Virginia Klema, and Gilbert W. Stewart. 1977. "Rosetak Document 4: Rank Degeneracies and Least Square Problems." SSRN Scholarly Paper. Rochester, NY.

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- How to pick the right ε?

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- How to pick the right ε?
- What if A and $A + N$ are nonnegative?

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